Day 5: Pascal's Triangle and the Binomial Theorem

As you expand binomials it becomes more and more tedious to calculate:

$$(x+y)^0 = 1$$

$$(x+y)^1 = x + y$$

$$(x+y)^2 = x^2 + 2xy + y^2$$

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x+y)^4 = \underbrace{x^4 + 4x^3y^1 + 6x^2y^2 + 4xy^3 + y^4}$$

$t_1 \quad t_2 \quad t_3 \quad t_4 \quad t_5$

But if we just look at the coefficients of each term, we can use a pattern:

$$
\begin{array}{lccccccccccccc}
(x+y)^0 = & & & & & & & 1 & & & & & & \\
(x+y)^1 = & & & & & & 1 & & 1 & & & & & \\
(x+y)^2 = & & & & & 1 & & 2 & & 1 & & & & \\
(x+y)^3 = & & & & 1 & & 3 & & 3 & & 1 & & & \\
(x+y)^4 = & & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
(x+y)^5 & & & 1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

Called Pascal's Triangle

1 6 15 20 15 6 1

Let's look at the following

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

- The sum of the exponents on any term equals the exponent on your binomial

$4x^3y^1$ $= (x+y)^4$

$3+1 = 4$

- The exponent on "y" is one less than the term number

$6x^2y^2 = t_3$

# of terms is always 1 more than the exponent on your binomial.

Eg 1) Find the fourth term in the expansion of $(x+y)^5$

Coefficient: 10

Variables: x^2y^3

$t_4 = 10x^2y^3$

$t_5 \rightarrow (x+y)^6$

Coefficient: 15

Variables: x^2y^4

$15x^2y^4$

It turns out that the coefficients found using Pascal's Triangle can also be generated using combinations:

$$(x+y)^0 = \quad {}_0C_0$$

$$(x+y)^1 = \quad {}_1C_0 \quad {}_1C_1$$

$$(x+y)^2 = \quad {}_2C_0 \quad {}_2C_1 \quad {}_2C_2$$

$$(x+y)^3 = \quad \underset{t_1}{{}_3C_0} \quad \underset{t_2}{{}_3C_1} \quad \underset{t_3}{{}_3C_2} \quad \underset{t_4}{{}_3C_3}$$

$\underset{\text{exponent}}{n}C_r = \underset{\text{binomial}}{{}_3}C_1 \leftarrow$ one less than the term #.

If we also consider that $(x+y)^2 = x^n y^0 + x^{n-1}y^1 + x^{n-2}y^2$ and combine this result with the above combinations we get the Binomial Theorem:

$$(x+y)^n = {}_nC_0x^n + {}_nC_1x^{n-1}y + {}_nC_2x^{n-2}y^2 + \ldots {}_nC_ny^n$$

This theorem can be used to expand any binomial (without even knowing further terms of Pascal's triangle). If you are using any other terms than x and y, simply replace them in the formula.

Eg 2) Find the 4^{th} term in $(x+y)^{9}$

Coefficient: ${}_9C_3 = 84$

Variables: x^6y^3

$84x^6y^3$

Eg 3) Determine the 5^{th} term in $(2a-b)^{8}$

$(x+y)^8$

C: ${}_8C_4 = 70$

V: x^4y^4

$70x^4y^4$

$70(2a)^4(-b)^4$

$70\cdot(16a^4)(1\cdot b^4)$

$1120a^4b^4$

$t_3 \rightarrow (3x-2)^7$

${}_7C_2(3x)^5(-2)^2$

$21(243x^5)(4)$

$20412x^5$

Eg 4) Write the expansion of $(2-3x)^{4}$

$(x+y)^4$

${}_4C_0x^4y^0 + {}_4C_1x^3y^1 + {}_4C_2x^2y^2 + {}_4C_3xy^3 + {}_4C_4y^4$

$1(2)^4 + 4(2)^3(-3x)^1 + 6(2)^2(-3x)^2 +$

$4(2)(-3x)^3 + 1(-3x)^4$

$16 - 96x + 216x^2 - 216x^3 + 81x^4$

Assignment

Handout - Pascal's Triangle ~~#1-5, 9-11~~ 1, 3, 5

Pg. 396 #4-6

odds